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By (1), $\{(\pi)_{0,0}\}[\alpha](0)$ is defined for each α. Let

$$R(\pi, a) \equiv (E\alpha)[a = \bar{\alpha}(x) \text{ for } x = (\{(\pi)_{0,0}\}[\alpha](0))_0].$$

Clearly **(5)** $(\beta)(Ex)R(\pi, \bar{\beta}(x))$.

We define a partial recursive predicate R_1 thus.

$$R_1(\pi, a) \simeq [a = \overline{\alpha_1}(x_1) \text{ for } \alpha_1 = \lambda t\,(a)_t \dot{-} 1,\ x_1 = (\{(\pi)_{0,0}\}[\alpha_1](0))_0].$$

We show that **(6)** $R(\pi, a) \equiv R_1(\pi, a)$. Assume $R(\pi, a)$, and put $a = \bar{\alpha}(x)$ with $x = (\{(\pi)_{0,0}\}[\alpha](0))_0$. By (1), $(\{(\pi)_{0,0}\}[\alpha])_{1,0}$ realizes-Ψ, α, x $\mathrm{R}(\bar{\alpha}(x))$, whence by Lemma 9.1 (a) it realizes-$\Psi, \bar{\alpha}(x)$ $\mathrm{R}(a)$. Let $\alpha_1 = \lambda t\,(a)_t \dot{-} 1$. Then α_1 agrees with α in its first x values, so $\overline{\alpha_1}(x) = \bar{\alpha}(x)$, and by Lemma 9.1 (a) $(\{(\pi)_{0,0}\}[\alpha])_{1,0}$ realizes-Ψ, α_1, x $\mathrm{R}(\bar{\alpha}(y))$. By (4) with α_1, x and $x_1 = (\{(\pi)_{0,0}\}[\alpha_1](0))_0$ as the α, y and x, $x = x_1$. So $a = \overline{\alpha_1}(x_1)$. Conversely, $a = \overline{\alpha_1}(x_1)$ implies $R(\pi, a)$.

We shall find a partial recursive function η with the following property. Let S_1^π be the set of the sequence numbers barred with respect to $\lambda a\, R_1(\pi, a)$ (cf. 6.3, 6.5, 6.6). **(7)** $a \in S_1^\pi \to \{\eta[\pi, a]$ realizes-Ψ, a $\mathrm{A}(a)\}$. To prove this, we use an intuitive application of the bar theorem, i.e. we use an induction over S_1^π (the informal analog of ×26.8a in 6.11, with the recursiveness of $\lambda a\, R_1(\pi, a)$ providing the first hypothesis). We begin by giving the basis and induction step. In each we derive a specification for $\eta[\pi, a]$ that will suffice there. Afterwards we show that a partial recursive η can be chosen to satisfy both specifications.

BASIS: $R_1(\pi, a)$. Then $a = \overline{\alpha_1}(x_1)$ etc. By (1), $(\{(\pi)_{0,0}\}[\alpha_1])_{1,0}$ realizes-Ψ, α_1, x_1 $\mathrm{R}(\bar{\alpha}(x))$, whence by Lemma 9.1 (a) it realizes-Ψ, a $\mathrm{R}(a)$. Also $\mathrm{Seq}(a)$, so $\lambda t\, 0$ realizes-a $\mathrm{Seq}(a)$. So using (2), $\eta[\pi, a]$ will realize-Ψ, a $\mathrm{A}(a)$ if $\eta[\pi, a] = \{\{(\pi)_{0,1}\}[a]\}[\lambda t\, 0, (\{(\pi)_{0,0}\}[\alpha_1])_{1,0}]$.

IND. STEP: $\mathrm{Seq}(a)\ \&\ (s)[a*2^{s+1} \in S_1^\pi]$. By hyp. ind., for each s, $\eta[\pi, a*2^{s+1}]$ realizes-$\Psi, a*2^{s+1}$ $\mathrm{A}(a)$, whence using Lemma 9.1 (a) it realizes-Ψ, a, s $\mathrm{A}(a*2^{s+1})$. So, using (3), $\eta[\pi, a]$ will realize-Ψ, a $\mathrm{A}(a)$ if $\eta[\pi, a] = \{\{(\pi)_1\}[a]\}[\lambda t\, 0, \Lambda s\, \eta[\pi, a*2^{s+1}]]$.

DEFINITION OF η. It will suffice to have $\eta[\pi, a] = \lambda u\, \eta(\pi, a, u)$ where

$$* \quad \eta(\pi, a, u) \simeq \begin{cases} \{\{(\pi)_{0,1}\}[a]\}[\lambda t\, 0, (\{(\pi)_{0,0}\}[\lambda t\, (a)_t \dot{-} 1])_{1,0}](u) & \text{if } R_1(\pi, a), \\ \{\{(\pi)_1\}[a]\}[\lambda t\, 0, \Lambda s\, \lambda t\, \eta(\pi, a*2^{s+1}, t)](u) & \text{otherwise.} \end{cases}$$

Upon replacing η by $\{z\}$, this equation assumes the form $\{z\}(\pi, a, u) \simeq \psi(z, \pi, a, u)$ with a partial recursive ψ. A solution e for z is given by

the recursion theorem IM p. 353 for π as the Ψ with uniformity. We take $\eta(\pi, a, u) \simeq \{e\}(\pi, a, u)$; as remarked after Lemma 8.1 (with (8.2a)), the specialization of z to e under the operation Λs is valid.

By (5) and (6): **(8)** $1 \in S_1^\pi$. Hence by (7), $\eta[\pi, 1]$ realizes-Ψ, 1 $\mathrm{A}(a)$, whence by Lemma 9.1 (a): **(9)** $\eta[\pi, 1]$ realizes-Ψ $\mathrm{A}(1)$.

Finally, $\Lambda\pi\, \eta[\pi, 1]$ realizes-Ψ the axiom.

AXIOM SCHEMA ×27.1. $\forall\alpha\exists\beta \mathrm{A}(\alpha, \beta) \supset \exists\tau\forall\alpha\{\forall t\exists! y\tau(2^{t+1}*\bar{\alpha}(y)) > 0\ \&\ \forall\beta[\forall t\exists y\tau(2^{t+1}*\bar{\alpha}(y)) = \beta(t)+1 \supset \mathrm{A}(\alpha, \beta)]\}$. Continuity

Assume (outside the definitions of the recursive functions and the final step) that π realizes-Ψ $\forall\alpha\exists\beta \mathrm{A}(\alpha, \beta)$.

Consider any α. Now **(1)** for each α, $(\{\pi\}[\alpha])_1$ realizes-Ψ, α, β_1 $\mathrm{A}(\alpha, \beta)$ for $\beta_1 = \{(\{\pi\}[\alpha])_0\}$. Let $\tau = \Lambda\alpha\, \beta_1 = \Lambda\alpha\, \{(\{\pi\}[\alpha])_0\}$. By (1) and Lemma 8.1, $\{\tau\}[\alpha]$ is properly defined, i.e. **(2)** $(t)(E!y)\tau(2^{t+1}*\bar{\alpha}(y)) > 0$, and $\{\tau\}[\alpha] = \beta_1$, whence by (8.1): **(3)** $(t)\tau(2^{t+1}*\bar{\alpha}(y_t)) = \beta_1(t)+1$ where $y_t = \mu y\tau(2^{t+1}*\bar{\alpha}(y)) > 0$.

Now we seek a function ρ_0 to realize-τ, α $\forall t\exists! y\tau(2^{t+1}*\bar{\alpha}(y)) > 0$, i.e. $\forall t\exists y[\tau(2^{t+1}*\bar{\alpha}(y)) > 0\ \&\ \forall z(\tau(2^{t+1}*\bar{\alpha}(z)) > 0 \supset y = z)]$, taking the equality as a prime formula (cf. preceding *15.1 in 5.5). Consider any t. Using (3), $\tau(2^{t+1}*\bar{\alpha}(y)) > 0$ is true-τ, α, t, y_t and hence is realized-τ, α, t, y_t by $\lambda s\, 0$. If σ realizes-τ, α, t, z $\tau(2^{t+1}*\bar{\alpha}(z)) > 0$, then $\tau(2^{t+1}*\bar{\alpha}(z)) > 0$ is true-τ, α, t, z, hence by (2) $z = y_t$, and hence $\lambda s\, 0$ realizes-z, y_t $z = y$. Combining these results, $\forall t\exists! y\tau(2^{t+1}*\bar{\alpha}(y)) > 0$ is realized-τ, α by $\rho_0 = \Lambda t\, \langle \mu y\tau(2^{t+1}*\bar{\alpha}(y)) > 0, \langle \lambda s\, 0, \Lambda z\, \Lambda\sigma\, \lambda s\, 0\rangle\rangle$.

Next we seek a function ρ_1 to realize-Ψ, τ, α $\forall\beta[\forall t\exists y\tau(2^{t+1}*\bar{\alpha}(y)) = \beta(t)+1 \supset \mathrm{A}(\alpha, \beta)]$. Consider any β. Suppose σ realizes-τ, α, β $\forall t\exists y\tau(2^{t+1}*\bar{\alpha}(y)) = \beta(t)+1$. Then, for each t, $(\{\sigma\}[t])_1$ realizes-$\tau, \alpha, \beta, \bar{y}_t$ $\tau(2^{t+1}*\bar{\alpha}(y)) = \beta(t)+1$ for $\bar{y}_t = (\{\sigma\}[t](0))_0$; thus $(t)\tau(2^{t+1}*\bar{\alpha}(\bar{y}_t)) = \beta(t)+1$. Hence by (2) and (3), $\beta = \beta_1$, so by (1) $(\{\pi\}[\alpha])_1$ realizes-Ψ, α, β $\mathrm{A}(\alpha, \beta)$. So we take $\rho_1 = \Lambda\beta\, \Lambda\sigma\, (\{\pi\}[\alpha])_1$.

Altogether, the axiom in question is realized-Ψ by $\Lambda\pi\, \langle \Lambda\tau, \Lambda\alpha \langle \rho_0, \rho_1\rangle\rangle$ for τ, ρ_0, ρ_1 as above.

(b) Say Γ is $\mathrm{D}_1, \ldots, \mathrm{D}_l$. Now each D_j has a realization function $\varphi_j[\Psi_j]$ general recursive in some finitely-many functions of T; let Σ be a list of all the functions of T thus used for $j = 1, \ldots, l$, reduced (if necessary) to one-place functions by end 8.2. By the penultimate remark in 8.2, it now suffices to construct, for each formula A_i of the given deduction, a realization function $\lambda\Psi_i\, \varphi_i[\Psi_i]$ of the form $\lambda\Psi_i\, \varphi_i[\Psi_i, \Sigma]$ with $\lambda\Psi_i\Sigma\, \varphi_i[\Psi_i, \Sigma]$ partial recursive; e.g. for Rule 9N we now use $\Lambda\gamma\, \Lambda x\, \{\psi[\Psi, x, \Sigma]\}[\gamma]$.

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α realizes-Ψ A; by Clause 4 in 8.5, we must infer from (1) that $\{\Lambda\alpha\,\Lambda\beta\,\alpha\}[\alpha]$ realizes-Ψ $B \supset A$. But by (8.3), $\{\Lambda\alpha\,\Lambda\beta\,\alpha\}[\alpha] = \Lambda\beta\,\alpha$. Suppose **(2)** β realizes-Ψ B; we must infer from (1) and (2) that $\{\Lambda\beta\,\alpha\}[\beta]$ realizes-Ψ A. By (8.3), $\{\Lambda\beta\,\alpha\}[\beta] = \alpha$; so what we need is (1).

1b. $(A \supset B) \supset ((A \supset (B \supset C)) \supset (A \supset C))$. Also 7, via Lemma 9.2. $\Lambda\pi\,\Lambda\rho\,\Lambda\alpha\,\{\{\rho\}[\alpha]\}[\{\pi\}[\alpha]]$.

3. $A \supset (B \supset A \,\&\, B)$. $\Lambda\alpha\,\Lambda\beta\,\langle\alpha, \beta\rangle$.

4a. $A \,\&\, B \supset A$. $\Lambda\gamma\,(\gamma)_0$. 4b. $A \,\&\, B \supset B$. $\Lambda\gamma\,(\gamma)_1$.

5a. $A \supset A \vee B$. $\Lambda\alpha\,\langle 0, \alpha\rangle$. 5b. $B \supset A \vee B$. $\Lambda\beta\,\langle 1, \beta\rangle$.

6. $(A \supset C) \supset ((B \supset C) \supset (A \vee B \supset C))$. $\Lambda\pi\,\Lambda\rho\,\Lambda\sigma$ $\lambda t\,\overline{\mathrm{sg}}((\sigma(0))_0)\cdot(\{\pi\}[(\sigma)_1])(t) + \mathrm{sg}((\sigma(0))_0)\cdot(\{\rho\}[(\sigma)_1])(t)$.

8^{I}. $\neg A \supset (A \supset B)$. $\Lambda\pi\,\lambda t\,0$. Suppose π realizes-Ψ $\neg$A. Then no function α realizes-Ψ A. So any function, e.g. $\lambda t\,0$, realizes-Ψ $A \supset B$.

10N. $\forall x A(x) \supset A(t)$, where A(x), t are as in Lemma 9.1 (a), so the free variables of the axiom are only Ψ, x. $\Lambda\pi\,\{\pi\}[t(\Psi, x)]$. For, suppose π realizes-Ψ, x $\forall x A(x)$. Then by Lemma 8.2, π realizes-Ψ $\forall x A(x)$. So $\{\pi\}[t(\Psi, x)]$ realizes-$\Psi, t(\Psi, x)$ A(x), whence by Lemma 9.1 (a) $\{\pi\}[t(\Psi, x)]$ realizes-Ψ, x A(t).

10F. $\forall\alpha A(\alpha) \supset A(u)$. $\Lambda\pi\,\{\pi\}[u[\Psi, \alpha]]$.

11N. $A(t) \supset \exists x A(x)$. $\Lambda\pi\,\langle t(\Psi, x), \pi\rangle$.

11F. $A(u) \supset \exists\alpha A(\alpha)$. $\Lambda\pi\,\langle \Lambda\,u[\Psi, \alpha], \pi\rangle$.

* 13. $A(0) \,\&\, \forall x(A(x) \supset A(x')) \supset A(x)$. $\Lambda\alpha\,\rho[x, \alpha]$, where ρ is defined by the "functional recursion"

$$\rho[0, \alpha] = (\alpha)_0,$$
$$\rho[x', \alpha] = \{\{(\alpha)_1\}[x]\}[\rho[x, \alpha]].$$

Writing $\rho[x, \alpha] = \lambda t\,\rho(x, \alpha, t)$, this takes the form

$$\rho(0, \alpha, t) = \psi(\alpha, t),$$
$$\rho(x', \alpha, t) \simeq \chi(x, \alpha, \lambda t\,\rho(x, \alpha, t), t)$$

where ψ is primitive, and χ is partial, recursive. To prove this ρ partial recursive, we apply the recursion theorem IM p. 353 for α as the Ψ $(l = 1)$ with uniformity to solve for z the equation

$$\{z\}(x, \alpha, t) \simeq \begin{cases} \psi(\alpha, t) \text{ if } x = 0, \\ \chi(x \dot{-} 1, \alpha, \lambda t\,\{z\}(x \dot{-} 1, \alpha, t), t) \text{ if } x \neq 0. \end{cases}$$

Call the solution e, and put $\rho(x, \alpha, t) = \{e\}(x, \alpha, t)$. (Cf. Lemma 3.2, Kleene 1956 § 4, 1959 XXIV.)

14, 17, ×1.1: $\Lambda\pi\,\lambda t\,0$. 16: $\Lambda\pi\,\Lambda\rho\,\lambda t\,0$.

15, and all prime axioms (namely, 18–21, ×0.1, the axioms of Group D): $\lambda t\,0$.

×2.1. $\forall x \exists\alpha A(x, \alpha) \supset \exists\alpha\forall x A(x, \lambda y \alpha(\langle x, y\rangle))$. $\Lambda\pi$ $\langle \Lambda\,\lambda t\,\{(\{\pi\}[(t)_0])_0\}((t)_1), \Lambda x\,(\{\pi\}[x])_1\rangle$. Suppose π realizes-Ψ $\forall x\exists\alpha A(x, \alpha)$. Then, for each x, $(\{\pi\}[x])_1$ realizes-$\Psi, x, \{(\{\pi\}[x])_0\}$ A(x, α). Hence by Lemma 9.1 (b), $(\{\pi\}[x])_1$ realizes-$\Psi, x, \lambda t\,\{(\{\pi\}[(t)_0])_0\}((t)_1)$ A(x, λy $\alpha(\langle x, y\rangle)$).

Rules of inference. 2. A, $A \supset B$ / B. Noting 8.7, we choose Ψ to include all variables free in $A \supset B$. By hyp. ind., there are general recursive functions α and ψ such that, for each Ψ, $\alpha[\Psi]$ realizes-Ψ A and $\psi[\Psi]$ realizes-Ψ $A \supset B$. Let $\varphi[\Psi] = \{\psi[\Psi]\}[\alpha[\Psi]]$. Then φ is partial recursive, and, for each Ψ, $\varphi[\Psi]$ realizes-Ψ B; hence φ is general recursive.

9N. $C \supset A(x)$ / $C \supset \forall x A(x)$. Say, for each Ψ and x, $\psi[\Psi, x]$ realizes-Ψ, x $C \supset A(x)$. Then, for each Ψ, $\Lambda\gamma\,\Lambda x\,\{\psi[\Psi, x]\}[\gamma]$ realizes-Ψ $C \supset \forall x A(x)$. 9F. $\Lambda\gamma\,\Lambda\alpha\,\{\psi[\Psi, \alpha]\}[\gamma]$.

12F. $A(\alpha) \supset C$ / $\exists\alpha A(\alpha) \supset C$. $\Lambda\pi\,\{\psi[\Psi, \{(\pi)_0\}]\}[(\pi)_1]$ is a realization function for the conclusion, if $\psi[\Psi, \alpha]$ is one for the premise. 12N. $\Lambda\pi\,\{\psi[\Psi, (\pi(0))_0]\}[(\pi)_1]$.

Axiom Schema ×26.3c. $\forall\alpha\exists! x R(\bar{\alpha}(x)) \,\&\, \forall a[\mathrm{Seq}(a) \,\&\, R(a) \supset A(a)] \,\&\, \forall a[\mathrm{Seq}(a) \,\&\, \forall s A(a * 2^{s+1}) \supset A(a)] \supset A(1)$. Bar Induction

Assume that π realizes-Ψ the antecedent of the main implication of an axiom by this schema containing free only Ψ; all the definitions and inferences below are under this assumption until the final step, except that, when we say a predicate or function with π as a variable is partial recursive, π ranges over all functions.

Now $(\pi)_{0,0}$ realizes-Ψ $\forall\alpha\exists! x R(\bar{\alpha}(x))$, i.e. $\forall\alpha\exists x[R(\bar{\alpha}(x)) \,\&\, \forall y(R(\bar{\alpha}(y)) \supset x{=}y)]$; $(\pi)_{0,1}$ realizes-Ψ $\forall a[\mathrm{Seq}(a) \,\&\, R(a) \supset A(a)]$; and $(\pi)_1$ realizes-Ψ $\forall a[\mathrm{Seq}(a) \,\&\, \forall s A(a * 2^{s+1}) \supset A(a)]$. Hence: **(1)** For each α, $(\{(\pi)_{0,0}\}[\alpha])_1$ realizes-Ψ, α, x $R(\bar{\alpha}(x)) \,\&\, \forall y(R(\bar{\alpha}(y)) \supset x{=}y)$ for $x = (\{(\pi)_{0,0}\}[\alpha](0))_0$. **(2)** For each a, ρ_0, ρ_1, if ρ_0 realizes-a Seq(a) and ρ_1 realizes-Ψ, a R(a), then $\{\{(\pi)_{0,1}\}[a]\}[\rho_0, \rho_1]$ realizes-Ψ, a A(a) (cf. (8.1c)). **(3)** For each a, ρ_0, ρ_1, if ρ_0 realizes-a Seq(a), and, for each s, $\{\rho_1\}[s]$ realizes-Ψ, a, s $A(a * 2^{s+1})$, then $\{\{(\pi)_1\}[a]\}[\rho_0, \rho_1]$ realizes-Ψ, a A(a).

Furthermore: **(4)** For each σ, α, y, if σ realizes-Ψ, α, y $R(\bar{\alpha}(y))$, then $y = x$ for the x of (1). For, by (1) $\{\{(\{(\pi)_{0,0}\}[\alpha])_{1,1}\}[y]\}[\sigma]$ realizes-x, y $x{=}y$, so $x{=}y$ is true-x, y.